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Loop operators and S-duality from curves on Riemann surfaces

Nadav Drukker,^a David R. Morrison^{b,c} and Takuya Okuda^d

^a*Institut für Physik, Humboldt-Universität zu Berlin,
Newtonstraße 15, D-12489 Berlin, Germany*

^b*Department of Mathematics, University of California,
Santa Barbara, CA 93106, U.S.A.*

^c*Department of Physics, University of California,
Santa Barbara, CA 93106, U.S.A.*

^d*Perimeter Institute for Theoretical Physics,
31 Caroline Street North, Waterloo, Ontario, N2L 2Y5, Canada*

E-mail: drukker@physik.hu-berlin.de, drm@physics.ucsb.edu,
takuya@perimeterinstitute.ca

ABSTRACT: We study Wilson-'t Hooft loop operators in a class of $\mathcal{N} = 2$ superconformal field theories recently introduced by Gaiotto. In the case that the gauge group is a product of $SU(2)$ groups, we classify all possible loop operators in terms of their electric and magnetic charges subject to the Dirac quantization condition. We then show that this precisely matches Dehn's classification of homotopy classes of non-self-intersecting curves on an associated Riemann surface — the same surface which characterizes the gauge theory. Our analysis provides an explicit prediction for the action of S-duality on loop operators in these theories which we check against the known duality transformation in several examples.

KEYWORDS: Duality in Gauge Field Theories, Supersymmetric gauge theory, AdS-CFT Correspondence

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1 Introduction

A new family of interacting four dimensional conformal field theories was recently presented in [1]. These theories can be motivated from several different points of view: by using building blocks taken from certain limits of quiver theories, in terms of brane webs [2], or in terms of a dimensional reduction of the six dimensional conformal field theory with $(2, 0)$ supersymmetry describing coincident M5-branes wrapped on a Riemann surface.

This Riemann surface plays an important rôle in all the different descriptions of the theory, as well as its AdS dual [3]. A four dimensional conformal field theory exists for any Riemann surface (allowing also for certain singularities). There is a one-to-one correspondence between the complex structures of the surface and the coupling constants of the gauge theory. More precisely, a closed surface of genus g corresponds to a theory with gauge group $SU(N)^{3g-3}$. Each $SU(N)$ factor can have its own coupling and theta angle which match the $3g - 3$ complex moduli of the surface.¹

In addition to the gauge fields (and their superpartners) these theories may include fundamental hypermultiplets as well as some mysterious strongly interacting conformal field theories which have been christened T_N . These T_N theories have a global $SU(N)^3$ flavor symmetry which will be generally gauged by some of the $SU(N)$ factors, coupling them to each other. The case where all gauge group factors are $SU(2)$ is rather special, as the T_2 theory is free.

Given that we can continuously deform these theories from weak to strong coupling, we would like to understand their behavior under S-duality. A particularly useful set of probes for analyzing this question are Wilson and 't Hooft loop operators and the dyonic

¹For more details on the case of punctured surfaces we refer the reader to the original papers [1, 3].

mixture of them. They should be mapped to each other under the action of S-duality. The purpose of this paper is to classify these operators and write down their transformation rules under S-duality; we find a particularly simple answer for theories with a gauge group which is a product of $SU(2)$ factors.

Very recently the question of the behavior of the partition function of these conformal field theories under S-duality was addressed in [4]. Interestingly, exactly in this case based on $SU(2)$ groups these authors found that the partition function of the four dimensional theory is equal to a correlation function of Liouville theory on the aforementioned Riemann surface.² We expect a generalization of their prescription to apply also to the calculation of the expectation values of loop operators. As we became aware of the great interest in understanding loop operators in these theories, we decided to publish our $SU(2)$ classification now, and defer to future work [6] some obvious open questions such as the analogue of these operators in Liouville theory, or the general case based on $SU(N)$ groups.

A loop operator may have an arbitrary shape in space-time, but in order to classify the possible *types* of operators it makes sense to choose a particular geometry for the curve. The simplest choices are a straight line or a circle (which are related by a conformal transformation). Locally, any smooth loop is approximately straight, and in these particular cases the loop operator can also preserve global supersymmetry.

As may be expected, after fixing the geometry of the loop in space-time, the remaining degrees of freedom are related to its gauge structure. Since this is intimately related to the associated Riemann surface [1], we expect to be able to classify loop operators in terms of the geometry of the surface. In fact, the classification turns out to be simple and beautiful: the loop operators are in one-to-one correspondence with non-self-intersecting curves on the surface (up to homotopy).³ To avoid confusion we repeat that the loops have a fixed geometry in space-time. The curves live on an auxiliary surface which is useful in order to classify these field theories.

This correspondence can then be used to understand the action of S-duality on the loop operators. It is believed that the S-duality group is isomorphic to the mapping class group of the surface [1, 7, 8]. The action of this group on the non-self-intersecting curves is quite complicated, but is well understood. This provides therefore a prediction for the action of the S-duality group on arbitrary Wilson-'t Hooft loop operators in the generalized quiver theories with $SU(2)$ gauge groups.

As motivation for the relation between curves and loop operators, it is useful to keep in mind the realizations of these field theories in M-theory. First, these theories arise on coincident M5-branes wrapping a Riemann surface. The loop operators correspond to M2-branes ending on the M5-branes along a 2-surface with one direction on the Riemann surface (and the other in the remaining four flat directions). For a large number of M5-

²A proposal for theories with $N > 2$ was given in [4, 5].

³The gauge theory endows the surface only with a complex structure. One can always find a surface with a hyperbolic metric and the same complex structure. Using this metric is sometimes convenient, as each homotopy class has a unique geodesic representative (and if the homotopy class can be represented by a non-self-intersecting curve, the geodesic is also non-self-intersecting). Therefore this matching applies also to the classification of non-intersecting geodesics with respect to the hyperbolic metric.

branes there is a dual description of this system within the AdS/CFT correspondence [3, 9]. In the dual geometry this Riemann surface also plays a rôle and the loop operators are again described by M2-branes with two directions inside AdS_5 and the third in the compact space. In the supergravity dual the supersymmetric embeddings are given by arbitrary geodesics on the Riemann surface with hyperbolic metric, allowing also for self-intersections (see appendix A). From the M-theory point of view the restriction to non-self-intersecting loops is quite mysterious and deserves further exploration.

The paper is organized as follows. In section 2, we illustrate our analysis by focusing on the prototypical conformal gauge theory with $\mathcal{N} = 2$ supersymmetry, namely the $SU(2)$ gauge theory with $N_F = 4$ flavors. Following this, section 3 discusses Wilson-'t Hooft loops in an arbitrary generalized quiver theory with $SU(2)$ gauge group factors characterized by a punctured Riemann surface. In section 4 we describe the classification of homotopy classes of curves with no self-intersection on the Riemann surface. The data used for the gauge theory and topological classifications are shown to be identical. Furthermore we discuss the transformation rules of curves under the action of the mapping class group. In cases where the S-duality transformations are explicitly known, we show that they agree with the action of the mapping class group. In other cases the geometric analysis provides a prediction for the action of S-duality. We summarize our results and present some further discussions in section 5.

In appendix A we analyze the supersymmetric embeddings of M2-branes in the Maldacena-Nuñez geometry. This is the supergravity analogue of the calculation in the body of the paper but in a very different regime, applicable for theories based on $SU(N)$ with large N , rather than $SU(2)$. This appendix can be read independently of the rest of the paper.

2 Prelude: $SU(2)$ gauge theory with $N_F = 4$

As a first example of a conformal $\mathcal{N} = 2$ theory with $SU(2)$ factors we consider the simple case of a single $SU(2)$ gauge group and $N_F = 4$ hypermultiplets in the fundamental representation. We want to analyze the supersymmetric loop operators of this theory. As mentioned in the introduction, the geometry of the loop will always be a straight line or a circle and the classification of operators corresponds to the charges they carry.

The most well-known loop operators are Wilson loops [10]. The supersymmetric generalization in $\mathcal{N} = 4$ super Yang-Mills theory was introduced in [11, 12]. Physically, these operators correspond to the insertion of an electrically charged BPS particle with infinite mass along the loop. The construction involves coupling the Wilson loop also to a real scalar field, which for supersymmetry should be in the same multiplet as the gauge field. In theories with extended $\mathcal{N} = 2$ supersymmetry the vector multiplet includes a complex scalar ϕ and we can take the real field $\text{Re}[\phi]$ (see [13]).

Wilson loop operators are defined as the trace of the holonomy along a loop⁴

$$\text{Tr}_R \mathcal{P} \exp \left[g_{\text{YM}} \oint (iA + \text{Re}[\phi] ds) \right], \quad (2.1)$$

⁴All fields have canonical kinetic terms and hence the explicit gauge coupling in the definition.

where the parameter s is normalized so that $|dx^\mu/ds|^2 = 1$. For a given geometry (for us a line or a circle) they are labeled by representations R of the gauge group G . In the case of a single $SU(2)$ this is a single positive half-integer spin $j = q/2$.

't Hooft loops are the magnetic counterpart of Wilson loops, and they insert probe monopoles along a loop in space-time. In the original definition by 't Hooft [14], the magnetic loop operators are classified by their topological charge. Kapustin introduced a finer classification [15] allowing for topologically trivial loops: a supersymmetric 't Hooft loop is defined by performing the path integral over field configurations with a specific singularity along the curve. Using spherical coordinates (r, ϑ, φ) in the space transverse to the time-like line the singularity near the loop takes the form⁵

$$g_{\text{YM}}A = \frac{\mu}{2}(1 - \cos \vartheta)d\varphi + \mathcal{O}(1/r), \quad g_{\text{YM}}\phi = -i\frac{\mu}{2r} + \mathcal{O}(1), \quad (2.2)$$

with

$$\mu = \begin{pmatrix} p/2 & 0 \\ 0 & -p/2 \end{pmatrix}. \quad (2.3)$$

The loop operators with odd p (which would be topologically non-trivial for an $SO(3)$ gauge group) can be defined only in a theory where none of the fields are charged under the center \mathbb{Z}_2 of the gauge group. Such is the case for a theory with matter only in the adjoint representation, but the case we consider here has four hypermultiplets in the fundamental representation, for which the usual Dirac quantization condition applies, meaning that p should be an even integer.

Supersymmetric dyonic loops are defined by inserting Wilson loops for the gauge group that remains unbroken in the 't Hooft loop background (2.2). For the Wilson loop carrying q units of electric flux we should rescale the singularity for the scalar field ϕ in (2.2) by $\sqrt{1 + q^2/p^2}/2$.⁶

Thus a general loop operator is specified by two integers: p , which is even, is the magnetic charge and q is the electric charge. Two pairs of charges which are related by the common Weyl group action

$$(p, q) \sim (-p, -q) \quad (2.4)$$

give identical loop operators. We may therefore assume $p \geq 0$ and for $p = 0$ we can take $q \geq 0$. As the simplest example, the Wilson loop in the fundamental representation has weights $(0, 1)$.

In anticipation of a later generalization, it is convenient to represent the gauge theory by a version of a quiver diagram shown in figure 1(a), where the internal edge is the $SU(2)$ gauge group and each open edge is a flavor group. The vertices are hypermultiplets in the fundamental of the both $SU(2)$ flavor groups as well as the gauge group.

⁵It is necessary to excite the scalar field in order to preserve some supersymmetries. The particular phase $-i$ is needed for the 't Hooft loop to preserve the same supercharges as the Wilson loop (2.1). This expression is modified for non-zero theta-angle.

⁶See [15] and [16] for more details.

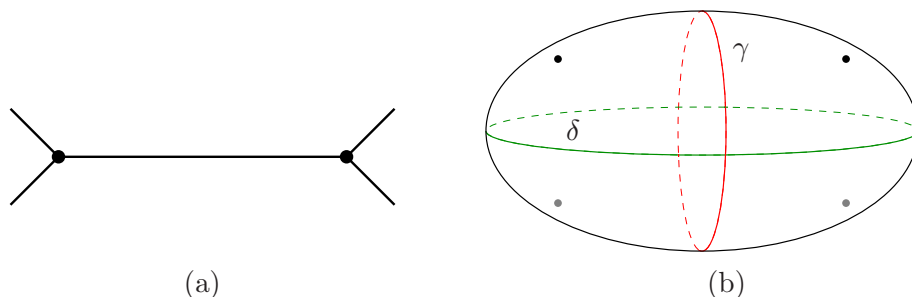


Figure 1. (a) A “quiver” diagram that represents the $SU(2)$ gauge theory with $N_F = 4$ flavors. (b) A sphere with four punctures obtained by fattening the quiver diagram. The geodesics we discuss are for the hyperbolic metric where each puncture has a 2π deficit angle and is at the end of an infinite tube. The red curve γ represents the Wilson loop in the fundamental representation, while the green curve δ corresponds to the minimal ’t Hooft loop.

It was observed in [7] that the parameter space of gauge couplings in this theory coincides with the Teichmüller space of a four-punctured sphere obtained by fattening the quiver diagram, as shown in figure 1(b).⁷

From this point of view, the quiver diagram figure 1(a) arises from a specific choice of “pants decomposition” of the four-punctured sphere which is natural at a corner of moduli space. The closed curve γ separates two punctures from the other two and each half of the sphere is topologically a “pair of pants” (or a three-punctured sphere). The curve δ gives another choice of quiver diagram describing the same gauge theory in a different S-duality frame.

In terms of this geometry we identify all supersymmetric loop operators with homotopy classes of non-self-intersecting curves on a four-punctured sphere as follows: closed curves without self-intersections are classified (homotopically) by the number of times they cross the curve γ and the twist they perform along γ . That is a pair of integers (p, q) where the number of crossings p is even and positive and q arbitrary (if $p = 0$, then the sign of q is ill defined and it will be taken to be positive).

For example, γ itself corresponds to $(0, 1)$, and the curve consisting of q copies of γ is represented by $(0, q)$. The curve δ is labeled by $(2, 0)$. By cutting the surface along γ twisting it and re-gluing, we get all curves with labels $(2, q)$. According to Dehn’s theorem explained in section 4, all homotopy classes with non-self-intersecting curves are classified by this labeling.

We see that Wilson-’t Hooft operators and non-self-intersecting curves are in one-to-one correspondence since they are both labeled by (p, q) with the same identification. As we will see in sections 3.1 and 4.1, this classification is consistent with the identification of the S-duality group $SL(2, \mathbb{Z})$ with the mapping class group of the four-punctured sphere.

⁷More precisely, the observation was that the parameter space modulo S-duality coincides with the complex structure moduli space of a sphere with four equivalent punctures. If we identify the mapping class group with the S-duality group, the two statements are equivalent.

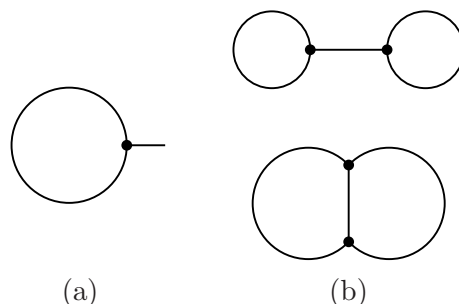


Figure 2. Examples of $SU(2)$ quiver diagrams. (a) The $\mathcal{N} = 2^*$ gauge theory (a mass deformation of $\mathcal{N} = 4$ SYM) corresponds to a once-punctured torus. (b) Two quiver gauge theories that are dual to each other. They correspond to a genus two surface with no punctures.

It is also natural to consider open curves connecting the punctures on the sphere. We discuss them in the next section after presenting the general case.

3 Classification of loop operators in gauge theory

As discussed earlier, new $\mathcal{N} = 2$ superconformal theories were discovered in [1]. These are “generalized quiver theories” which involve the conformal theories denoted by T_N in [3], as well as $SU(N)$ gauge groups with various values of N . For $N > 2$, T_N is an exotic theory without a known Lagrangian description, while T_2 is simply a free theory that contains four hypermultiplets, or equivalently eight half-hypermultiplets. The theory has flavor symmetry group $SU(2) \times SU(2) \times SU(2)$ under which all fields transform in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ representation.

In this paper we focus on a subclass of generalized quiver theories which are based on $SU(2)$ gauge groups and T_2 . Such a theory is represented by a generalized quiver diagram built from trivalent vertices connected by edges. See figure 1(a) as well as figure 2 for examples. An internal edge represents an $SU(2)$ factor in the gauge group whereas each T_2 theory corresponds to a trivalent vertex. The external (open) edges correspond to an $SU(2)$ flavor symmetry.

By thickening the edges, we can associate to each quiver diagram the topology of a Riemann surface. An open leg of a trivalent vertex corresponds to a puncture on the surface. Let g denote the genus of the surface and n the number of punctures.⁸ Then the theory has gauge group

$$G = SU(2)^{3g-3+n} = \prod_{j=1}^{3g-3+n} SU(2)_j \tag{3.1}$$

and flavor symmetry group

$$SU(2)^n = \prod_{j=3g-2+n}^{3g-3+2n} SU(2)_j. \tag{3.2}$$

⁸The number of trivalent vertices is $2g - 2 + n$, the number of internal edges is $3g - 3 + n$ and the number of open edges is n .

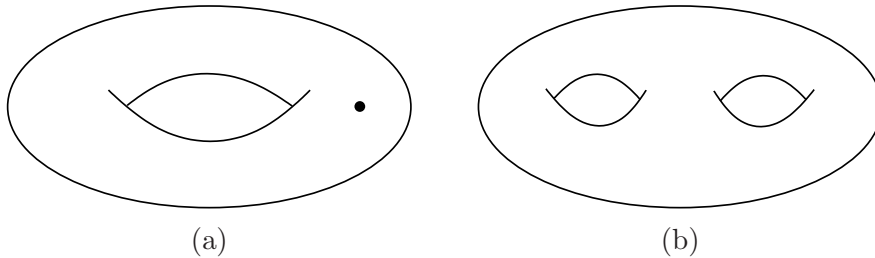


Figure 3. (a) A once-punctured torus. (b) A genus two Riemann surface. Two S-dual realizations of the theory based on this surface are given in figure 2(b).

Our aim is to classify Wilson-'t Hooft loop operators in this theory. In section 2 we presented these objects for a theory whose gauge group is a single $SU(2)$. For a general gauge group the Wilson loops are again given by (2.1), where now R is an arbitrary representation of the gauge group. The representation R may be specified by their highest weights $\nu \in \Lambda_w$, where Λ_w is the weight lattice. Supersymmetric 't Hooft loops are given again by (2.2) where μ generally takes values in the coweight lattice Λ_{cw} defined as the dual of the root lattice.

The most general supersymmetric loop operator will be dyonic, and this requires first choosing a magnetic source and adding on top an arbitrary Wilson loop in the gauge group left unbroken by the magnetic background.⁹ It was argued in [15] that general supersymmetric Wilson-'t Hooft operators are labeled by a pair of magnetic and electric weights up to identification by the action of the Weyl group:

$$(\mu, \nu) \in \Lambda_{cw} \times \Lambda_w, \quad (\mu, \nu) \sim (w \cdot \mu, w \cdot \nu), \quad w \in \mathcal{W}. \quad (3.3)$$

As mentioned for the example in section 2, in the presence of matter fields charged under the center of the gauge group, some restrictions need to be imposed on the allowed magnetic weights. The singular background (2.2) has a Dirac string along $\vartheta = \pi$ (where ϑ is an angular variable in the transverse space). When one goes around the Dirac string, a matter field Φ transforms as

$$\Phi \rightarrow e^{2\pi i \mu \cdot} \Phi, \quad (3.4)$$

where the exponential acts on Φ according to the representation in which Φ transforms. The Dirac string is thus invisible if and only if $\exp(2\pi i \mu \cdot)$ acts trivially on all matter fields. The set of allowed magnetic weights μ that satisfy this Dirac condition forms a sublattice of Λ_{cw} .

This analysis of Wilson-'t Hooft operators applies to any $\mathcal{N} = 2$ gauge theory. We now wish to apply it to the specific theories we are considering, where the gauge group is a product of $SU(2)$ factors. The general weights parameterizing the loop operator are the set of integers

$$(p_1, p_2, \dots, p_{3g-3+n}; q_1, q_2, \dots, q_{3g-3+n}). \quad (3.5)$$

⁹See [15] and [16] for a more precise definition of a dyonic loop operator.

The Weyl group $\mathcal{W} = (\mathbb{Z}_2)^{3g-3+n}$ acts by $p_j \rightarrow -p_j, q_j \rightarrow -q_j$ for any j and sets of integers related in this way specify the same operator. The definition of this operator involves two steps. First we demand that the path-integral be over singular field configurations satisfying near the loop¹⁰

$$\begin{aligned}
 g_{\text{YM}}^{(j)} A^{(j)} &= \begin{pmatrix} p_j/2 & 0 \\ 0 & -p_j/2 \end{pmatrix} \frac{1 - \cos \vartheta}{2} d\varphi + \mathcal{O}(1), \\
 g_{\text{YM}}^{(j)} \phi^{(j)} &= -\frac{i}{2r} \begin{pmatrix} \omega_j/2 & 0 \\ 0 & -\omega_j/2 \end{pmatrix} + \mathcal{O}(1).
 \end{aligned}
 \tag{3.6}$$

If p_j is non-zero, $\text{SU}(2)_j$ is broken to its maximal torus $\text{U}(1)$ by the background fields in (3.6). Second the path-integral is performed with the insertion of a Wilson loop specified by q_j for the unbroken gauge group. The coupling ω_j will be chosen so that the loop operator preserves the same supercharges as the pure Wilson loop (2.1). Though this definition is sufficient for classification purposes, a more precise one should include regularization and boundary terms.

Supercharges preserved by a loop operator depend on its magnetic charge, its electric charge and in addition the choice of ω_j . Thus ω_j should be adjusted independently for each $\text{SU}(2)_j$ such that the supercharges are shared by the full operator. A heuristic way to determine the value of ω_j is by considering the classical field configuration produced by the dyonic operator with weights (p_j, q_j) . Generalizing the treatment of the Abelian case in [15], the gauge potential and scalar field take the form

$$\begin{aligned}
 g_{\text{YM}} A_{\text{cl}} &= i \begin{pmatrix} q_j/2 & 0 \\ 0 & -q_j/2 \end{pmatrix} \frac{dt}{2r} + \begin{pmatrix} p_j/2 & 0 \\ 0 & -p_j/2 \end{pmatrix} \frac{1 - \cos \vartheta}{2} d\varphi, \\
 g_{\text{YM}} \phi_{\text{cl}} &= -\frac{i}{2r} \begin{pmatrix} \omega_j/2 & 0 \\ 0 & -\omega_j/2 \end{pmatrix}.
 \end{aligned}
 \tag{3.7}$$

The gaugino variation vanishes when $|\omega_j| = \sqrt{p_j^2 + q_j^2}$. In general we get a similar condition for each $\text{SU}(2)$ factor and they all have to be consistent, so that if for a purely electric loop operator we take a real scalar field with $\omega_j = i$ then for a pair of magnetic and electric weights (p_j, q_j) , we find $\omega_j = p_j + iq_j$. Thus we require that for all j

$$\omega_j = p_j + iq_j.
 \tag{3.8}$$

It turns out that in the particular class of quiver theories we are considering, it is rather natural to generalize the notion of Wilson-'t Hooft loop operators. This is done by introducing non-dynamical gauge and scalar fields $\hat{A}^{(j)}$ and $\hat{\phi}^{(j)}$ for the flavor symmetry groups $\text{SU}(2)_j$ and setting them to

$$\hat{A}^{(j)} = \begin{pmatrix} p_j/2 & 0 \\ 0 & -p_j/2 \end{pmatrix} \frac{1 - \cos \vartheta}{2} d\varphi, \quad \hat{\phi}^{(j)} = -\frac{i}{2r} \begin{pmatrix} p_j/2 & 0 \\ 0 & -p_j/2 \end{pmatrix},
 \tag{3.9}$$

¹⁰For simplicity we wrote the expression for the case when the complexified gauge coupling $\tau = \theta/2\pi + 4\pi i/g_{\text{YM}}^2$ is purely imaginary. A generalization exists for any theta angle.

for $j = 3g - 2 + n, \dots, 3g - 3 + 2n$.

We are using the same notation p_j both for dynamical and non-dynamical fields since this will allow us to treat them uniformly. All dynamical fields charged under the flavor groups $SU(2)_j$ couple to these background gauge fields and therefore are sections of appropriate vector bundles. The non-dynamical scalar is required if we want to preserve the supersymmetry of the hypermultiplets it couples to.

One reason the excitation of non-dynamical fields is natural is that they arise when we consider these theories as limits of theories with additional $SU(2)$ gauge factors, which reduce to flavor groups in the decoupling limit. If there was a non-trivial bundle, this remains in the decoupling limit. Due to that, it is also natural to identify non-dynamical field configurations related by the (flavor) Weyl group action $p_j \rightarrow -p_j$. In any case, since now the Weyl group is a global symmetry, its action can be read from the way the fields are charged under it (and can also be absorbed by a field redefinition).

The introduction of such non-dynamical gauge fields is a deformation of the theory rather than an operator. Such a deformation, combined with the Wilson-'t Hooft operator above, defines a generalized Wilson-'t Hooft loop. Our discussion so far implies that generalized Wilson-'t Hooft loops are labeled by $6g - 6 + 3n$ integers

$$(p_1, p_2, \dots, p_{3g-3+2n}; q_1, q_2, \dots, q_{3g-3+n}) \tag{3.10}$$

subject to identification by the (independent) Weyl group actions $(p_j, q_j) \rightarrow (-p_j, -q_j)$ for $j = 1, 2, \dots, 3g - 3 + n$ as well as $p_j \rightarrow -p_j$ for $j = 3g - 2 + n, \dots, 3g - 3 + 2n$. Ordinary Wilson-'t Hooft loop operators correspond to generalized weights such that $p_{3g-2+n} = \dots = p_{3g-3+2n} = 0$.

Let us now revisit the Dirac condition on the set of allowed Wilson-'t Hooft loops discussed above. Each trivalent vertex has matter fields which transform in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ representation under $SU(2)_i \times SU(2)_j \times SU(2)_k$, where each of i, j and k labels either an edge between two vertices or an open leg attached to the trivalent vertex. We allow the possibility that two of i, j and k are identical. When one transports such a matter field Φ around a Dirac string in the background fields (3.6) and (3.9), it gets transformed by the three matrices in the defining representations

$$\Phi \rightarrow \begin{pmatrix} e^{\pi i p_i} & 0 \\ 0 & e^{-\pi i p_i} \end{pmatrix} \otimes \begin{pmatrix} e^{\pi i p_j} & 0 \\ 0 & e^{-\pi i p_j} \end{pmatrix} \otimes \begin{pmatrix} e^{\pi i p_k} & 0 \\ 0 & e^{-\pi i p_k} \end{pmatrix} \cdot \Phi. \tag{3.11}$$

For Φ to be single-valued, we thus require that

$$p_i + p_j + p_k \in 2\mathbb{Z}. \tag{3.12}$$

This leads to the main result of this section. Generalized Wilson-'t Hooft loops in the $\mathcal{N} = 2$ conformal generalized quiver theory, corresponding to a Riemann surface of genus g with n punctures, are labeled by the magnetic and electric weights

$$(p_j, q_j) \equiv (p_1, p_2, \dots, p_{3g-3+2n}; q_1, q_2, \dots, q_{3g-3+n}) \tag{3.13}$$

where due to the action of the Weyl group we may assume that $p_j \geq 0$ and for any j such that $p_j = 0$, we can take $q_j \geq 0$. In addition, for any trivalent vertex, the sum of the three magnetic weights in the groups attached to it has to be even.¹¹

S-duality should act on these parameters generalizing the familiar exchange of Wilson and 't Hooft loops in $\mathcal{N} = 4$ super Yang-Mills theory [15]. In the next section we identify this classification with that of non-self-intersecting geodesics on Riemann surfaces with hyperbolic metrics. These geodesics transform in a computable way under action of the mapping class group, providing an explicit prediction for the action of S-duality on these Wilson-'t Hooft operators.

3.1 Examples

I. $\mathcal{N} = 4/\mathcal{N} = 2^*$ super Yang-Mills with $G = \text{SU}(2)$.

The quiver in figure 2(a) and the torus with one puncture in figure 3(a) represent the $\mathcal{N} = 4$ super Yang Mills theory with gauge group $\text{SU}(2)$, and an extra decoupled hypermultiplet. Alternatively, with the inclusion of a mass term for an adjoint hypermultiplet, this quiver represents the $\mathcal{N} = 2^*$ theory. Generalized Wilson-'t Hooft loops are labeled by three integers $(p_1, p_2; q_1)$, where ordinary loop operators correspond to the case $p_2 = 0$.

This theory is expected to have the same S-duality symmetry as the $\mathcal{N} = 4$ theory. Under S-duality the gauge coupling $\tau = \theta/2\pi + 4\pi i/g^2$ transforms as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \quad (3.14)$$

When the non-dynamical gauge field is turned off, p_2 vanishes and the loop operators transform, according to [15], as

$$(p, q) \mapsto (p, q) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (3.15)$$

In section 4.1 we give an explicit prediction for the transformation rules of the loop operators also with the non-dynamical gauge fields.

II. One $\text{SU}(2)$ group with $N_F = 4$.

This is the case considered in section 2. The genus is $g = 0$ and the number of punctures is $n = 4$ so there is a single edge and two trivalent vertices, each with two open legs. See figures 1(a) and (b). In the discussion there we did not include the non-dynamical fields, so there was a single magnetic weight $p_1 \equiv p$ which had to be even.

If we include the non-dynamical gauge fields $\hat{A}^{(2)}$, $\hat{A}^{(3)}$ coupling to the first trivalent vertex and $\hat{A}^{(4)}$ and $\hat{A}^{(5)}$ to the other one, then we have four more positive integers p_2, \dots, p_5 . Now p_1 may be odd, but both $p_1 + p_2 + p_3$ and $p_1 + p_4 + p_5$ should be

¹¹When a pair (p_j, q_j) are not relatively prime, the same set of charges matches also those of a (reducible) product of operators. We find it natural to view the matching as applied to the irreducible representation.

even. There is still only one electric weight $q_1 \equiv q$, which is an arbitrary integer (unless $p_1 = 0$ in which case $q_1 \geq 0$).

It is interesting to consider S-duality in this theory, which exchanges the rôles of γ and δ in figure 1(b) and rotates the quiver diagram in figure 1(a) by 90 degrees [1]. The S-duality group, which is isomorphic to $SL(2, \mathbb{Z})$, acts on the gauge coupling $\tau = \theta/2\pi + 4\pi i/g^2$ as

$$\tau \mapsto \frac{a\tau + b/2}{2c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \quad (3.16)$$

The transformation $\theta \rightarrow \theta + \pi$ is a duality of this theory because amplitudes with odd instanton numbers vanish due to fermionic zero-modes. This is consistent with the fact that the Riemann surface is invariant under the twist by angle π along the geodesic γ .

As for the transformations of the loop operators, in the case where the non-dynamical gauge fields are turned off, this problem was studied in [17]. Based on the transformation of electric and magnetic charges carried by BPS particles that can be inferred from the Seiberg-Witten curve, it was argued that the S-duality group acts on the Wilson-'t Hooft operators by right multiplication by the inverse matrix

$$(p, q) \mapsto (p, q) \begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}. \quad (3.17)$$

This in particular implies that under the generators

$$S = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad (3.18)$$

the weights transform as

$$S : (p, q) \rightarrow (-2q, p/2), \quad T : (p, q) \rightarrow (p, q - p/2). \quad (3.19)$$

The action of T is the Witten effect for loop operators in this theory, which shifts the electric weight by a multiple of the magnetic weight under the shift of the θ angle [18, 19]. We will compare the transformations of weights with the transformations of geodesics in section 4.1. The transformation rules of open curves provide a prediction for the action of S-duality on the operators with non-dynamical gauge fields.

III. Quiver theories for genus two surface.

Let us consider gauge theories with gauge group $SU(2)^3$ and no global symmetries. They can be represented as quiver diagrams with two vertices and three edges. Two examples of such theories are shown in figure 2(b). In the top quiver the matter fields transform in the fundamental of the gauge group represented by the central edge and as the bi-fundamental of one of the other two gauge groups. In the bottom quiver all matter fields are charged under all three gauge groups. The two theories

are characterized by a single Riemann surface of genus two with no punctures shown in figure 3, and thus should be related by S-duality.

In the first case, the most general supersymmetric loop operator is given by the set of integers

$$(p_j; q_j) \equiv (p_1, p_2, p_3; q_1, q_2, q_3) \tag{3.20}$$

up to the usual identification under the Weyl group. Taking the index 1 to refer to the edge in the middle, we find the condition

$$p_1 + 2p_2 \in 2\mathbb{Z}, \quad p_1 + 2p_3 \in 2\mathbb{Z}, \quad \Leftrightarrow \quad p_1 \in 2\mathbb{Z}. \tag{3.21}$$

In the second theory, the six integers $(p'_j; q'_j)$ that label loop operators are subject to the condition

$$p'_1 + p'_2 + p'_3 \in 2\mathbb{Z}. \tag{3.22}$$

Apart for the expectation that S-duality is the same as maps of the genus 2 surface, nothing more is known about S-duality for these theories. Our analysis in the next section provides an explicit conjecture for the S-duality action on the Wilson-'t Hooft operators.

4 Curves on punctured Riemann surfaces

As mentioned in the introduction and the preceding section, each of the $\mathcal{N} = 2$ theories we discuss is related to a punctured Riemann surface. From the M-theory descriptions of loop operators in these theories we are motivated to look at curves on this Riemann surface. A loop operator will arise from an M2-brane ending on a collection of M5-branes along a two dimensional manifold with one direction in space-time and the other on the Riemann surface. In appendix A we study the supergravity duals of the theories based on $SU(N)$ at large N , where the curves turn out to be geodesics with respect to the hyperbolic metric on the Riemann surface.

From the gauge theory side we saw that the quiver in figure 1 corresponds to a specific representation of the Riemann surface in terms of “pairs of pants” (see below). In fact a theorem due to Dehn classifies homotopy classes of non-self-intersecting curves on Riemann surfaces using exactly this representation of the Riemann surface. As we shall see, this classification matches perfectly that of the loop operators in the previous section.

We begin by describing the topology of an oriented punctured Riemann surface and the Fenchel-Nielsen coordinates for Teichmüller space, following [20] and [21]. For any (oriented) punctured Riemann surface Σ of negative Euler characteristic with genus g and n punctures, there exist $3g - 3 + n$ pairwise disjoint connected curves without self-intersection, none of which is homotopic to zero or to a curve arbitrarily close to a puncture, such that the complement of the curves is a union of $2g - 2 + n$ pairs-of-pants. We treat

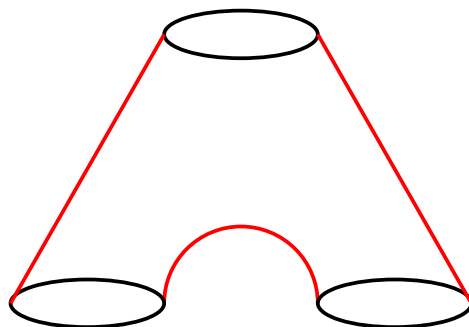


Figure 4. A pair of pants with seams

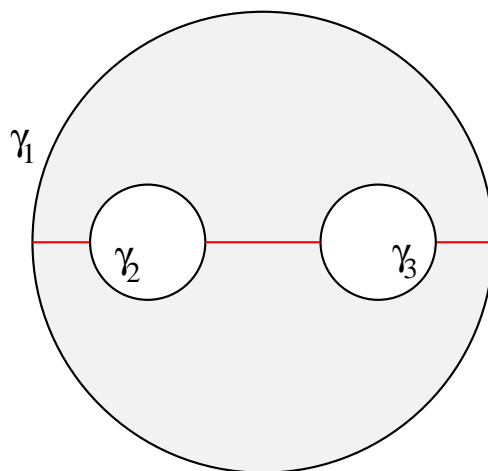


Figure 5. A pair of pants, shown as a disk with two holes

a puncture as a pants-leg of zero length, so some of these pairs-of-pants may look like a punctured annulus or a twice-punctured disk.

This decomposition is topological, but if a hyperbolic metric has been specified on Σ , then the curves giving the decomposition can be chosen to be geodesics (after a homotopy), and the resulting pairs of pants inherit constant curvature metrics. The lengths of the separating geodesics are determined by the metric; on the other hand, once the lengths of the three pants-legs have been specified, a pair-of-pants has a unique metric of constant curvature -1 , and there is a unique geodesic arc (the “seam”) connecting each pair of pants-legs, as shown in figure 4. An alternate visualization is given in figure 5

To re-assemble such a collection of pairs-of-pants into a punctured Riemann surface with metric (assuming that we have fixed in advance the set of pants-legs which have length zero and correspond to punctures), we need to specify both the length $\ell_j \in \mathbb{R}^+$ and a twist parameter $\theta_j \in \mathbb{R}$ for each pants-leg of non-zero length. The twist parameter measures the relative angular twist when gluing together the two pairs-of-pants along a common pants-leg, *i.e.*, the displacement of the seams. (Note that although twisting by an

integer multiple of 2π gives a diffeomorphic surface, the diffeomorphism is not isotopic to the identity; this diffeomorphism is known as a *Dehn twist*.) Fenchel and Nielsen showed that this assignment establishes an isomorphism between the associated Teichmüller space and $(\mathbb{R}^+)^{3g-3+n} \times \mathbb{R}^{3g-3+n}$.

There are many possible choices of pants decomposition (and of the corresponding Fenchel-Nielsen coordinates); we fix one such and use it to describe all homotopy classes of closed curves on Σ without self-intersection, as well as homotopy classes of arcs connecting punctures without self-intersection. The result is known as Dehn’s theorem.¹² We do not assume that the curve is connected; we *do* assume that no component of the curve is homotopic to zero, and that no component of the curve is homotopic to a curve arbitrarily close to one of the punctures.

If a hyperbolic metric has been specified on the Riemann surface, then each homotopy class of curves or arcs without self-intersection contains a unique geodesic (which also is without self-intersection). Dehn’s theorem can therefore be viewed as a classification of non-self-intersecting geodesics, rather than a classification of homotopy classes of non-self-intersecting curves.

The first step in Dehn’s analysis is to characterize collections of arcs without self-intersection on a single pair-of-pants. Dehn showed that any such collection of arcs can be moved, by a homotopy that leaves the endpoints of the arcs on the boundary, to a collection of arcs whose boundary points lie in the upper semicircles of each of the boundaries, and each of which is homotopic to one of six specific “basic” arcs, illustrated in figure 6.

Such a collection of arcs determines three non-negative integers $p_1, p_2,$ and p_3 which tell how many endpoints there are on each boundary circle $\gamma_1, \gamma_2,$ and γ_3 , subject to the condition that $p_1 + p_2 + p_3$ is even.

Conversely, given any three non-negative integers $p_1, p_2,$ and p_3 whose sum is even, there is a collection of non-intersecting arcs of the six basic types, unique up to homotopy, with the desired numbers of endpoints. To see this, first note that if $p_i > p_j + p_k$ and $p_j > p_i + p_k$ then $p_i + p_j > p_i + p_j + 2p_k$ so that p_k is negative, a contradiction. Thus, at most one of the numbers $p_i - p_j - p_k$ is positive.

If $p_i > p_j + p_k$, then we can use

$$\frac{1}{2}(p_i - p_j - p_k)\ell_{ii} + p_j\ell_{ij} + p_k\ell_{ik}. \tag{4.1}$$

On the other hand, if $p_j + p_k \geq p_i$ for all permutations of $\{1, 2, 3\}$, then we can use

$$\frac{1}{2}(p_1 + p_2 - p_3)\ell_{12} + \frac{1}{2}(p_1 + p_3 - p_2)\ell_{13} + \frac{1}{2}(p_2 + p_3 - p_1)\ell_{23}. \tag{4.2}$$

The second step in Dehn’s theorem is to attach the endpoints on either side of a boundary circle. One possibility is that the given non-self-intersecting curve has intersection number 0 with γ_j so that no attachment needs to be done; in this case, we define the *twisting number* q_j to be the number of components of the curve which are homotopic to γ_j . (Thus, $p_j = 0$ implies $q_j \geq 0$.)

¹²This theorem was found by Dehn in the 1920’s (but only published much later [22]), and rediscovered by Thurston in the 1970’s (also published much later [23]). We follow the account in [24].

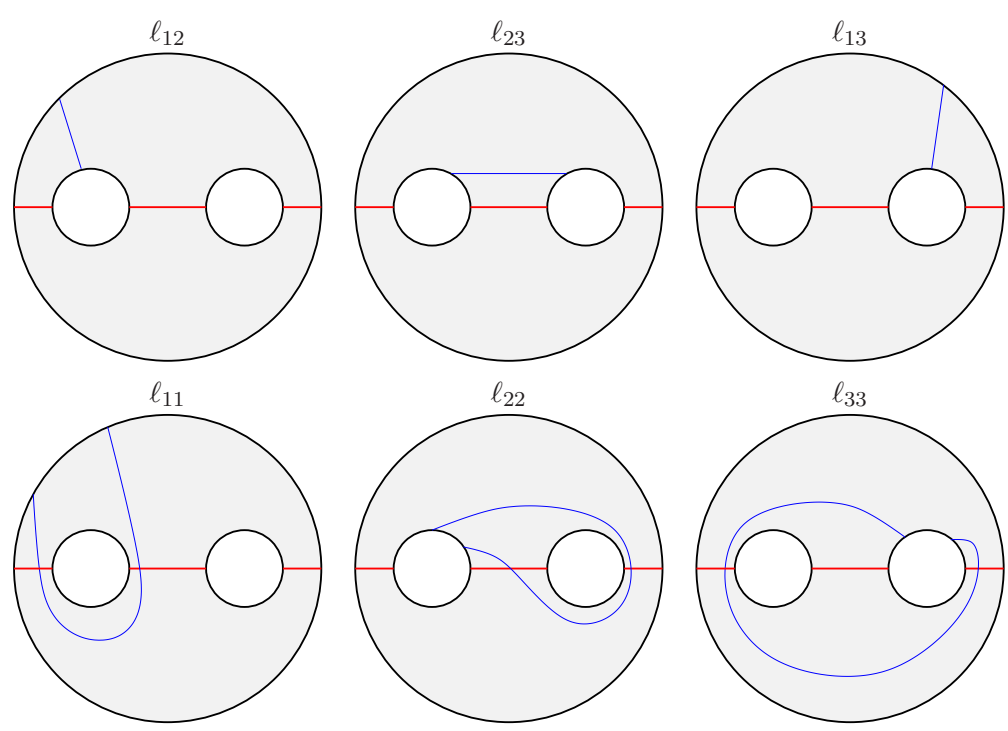


Figure 6. Basic arcs on a pair-of-pants

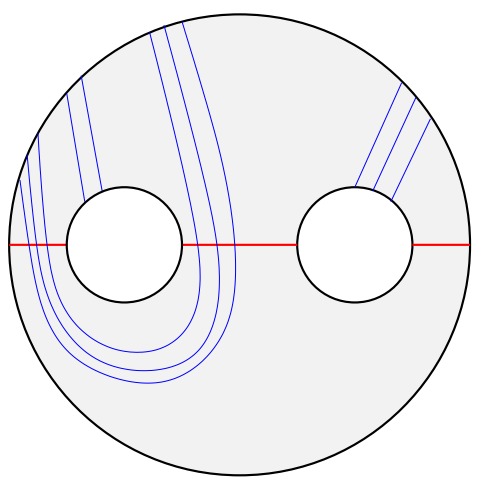


Figure 7. Example of arcs: $p_1 = 11, p_2 = 2, p_3 = 3$

Otherwise, when $p_j \neq 0$, there is a canonical gluing of the curves along the boundary γ_j , since using the basic arcs in figure 6, the p endpoints are on one half of the boundary circle. This gluing is the one labeled by $q_j = 0$. As mentioned, it is possible to twist the curves before gluing, which means that this identification will not be obeyed. After a homotopy, the twisting can be confined to a small annulus containing the curve, as illustrated in figure 8. The twisting number q_j is positive if the attaching curves bend to the right (as in the figure), and is negative if the attaching curves bend to the left. The

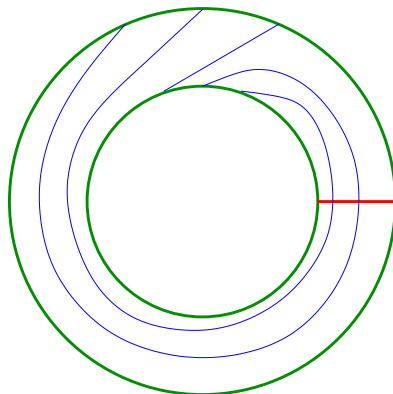


Figure 8. Example of twisting: intersection number $p = 3$, twisting number $q = +2$.

magnitude $|q_j|$ of the twisting number counts the number of shifts when matching the arcs on the two sides. A simple way to calculate $|q_j|$ is to count the number of intersections of the curve with a fixed arc which connects the two boundaries of the annulus (as illustrated in figure 8). Note that if we make a Dehn twist along γ_j (in the direction which curves to the right), the twisting number changes by $q_j \mapsto q_j + p_j$.

The intersection number p_j and the twisting number q_j give a complete local description of how the given curve behaves near γ_j . For the local behavior near each puncture, we specify a corresponding intersection number p_j (the number of endpoints at the puncture, or the intersection with a small loop near the puncture) but do not need to specify a twisting number q_j , since there is no winding at the puncture (up to homotopy).

The precise statement of Dehn’s theorem involves intersection numbers between non-self-intersecting curves. Since the topological version of the classification involves homotopy classes of curves rather than specific curves, it is natural to define $\#(\gamma \cap \delta)$ to be the minimum of the number of intersection points as γ and δ vary among non-self-intersecting curves in their respective homology classes. (Note that all intersections are counted *positively* — there is no sign or orientation taken into account.) This definition becomes much simpler when Σ has been given a hyperbolic metric: it turns out that one geodesic from each of the two homotopy classes can be used to compute the intersection number $\#(\gamma \cap \delta)$ directly, just by counting intersection points of the geodesics.

Dehn’s theorem. Let Σ be an oriented punctured Riemann surface of negative Euler characteristic with genus g and n punctures. Let $\gamma_1, \dots, \gamma_{3g-3+n}$ be pairwise disjoint connected curves without self-intersection whose complement is a pants decomposition of Σ , and let $\gamma_{3g-3+n+1}, \dots, \gamma_{3g-3+2n}$ be simple closed curves near the punctures. Define a mapping

$$\gamma \mapsto (\#(\gamma \cap \gamma_j); q_j) \in (\mathbb{Z}_{\geq 0})^{3g-3+2n} \times \mathbb{Z}^{3g-3+n} \tag{4.3}$$

which assigns to each homotopy class of closed curves without self-intersection or arcs connecting punctures without self-intersection its intersection number $p_j = \#(\gamma \cap \gamma_j)$ with γ_j , $1 \leq j \leq 3g - 3 + 2n$ and its twisting number q_j with respect to γ_j , $1 \leq j \leq 3g - 3 + n$.

(Note that these intersection and twisting numbers depend only on the homotopy class of γ .) Then this mapping is one-to-one, and its image is

$$\begin{aligned} & \{(p_1, p_2, \dots, p_{3g-3+2n}; q_1, q_2, \dots, q_{3g-3+n}) \\ & \quad | \text{ if } p_j = 0 \text{ then } q_j \geq 0, \text{ and } p_i + p_j + p_k \in 2\mathbb{Z} \\ & \quad \text{whenever } \gamma_i \cup \gamma_j \cup \gamma_k \text{ is the boundary of a pair-of-pants}\}. \end{aligned}$$

The integers p_j, q_j are called the *Dehn-Thurston parameters* of γ .

It is easy to see that Dehn’s classification of homotopy classes of non-self-intersecting curves (or of non-self-intersecting geodesics when a hyperbolic metric is being used) is in one-to-one correspondence with the classification of Wilson-’t Hooft loop operators in the gauge theory in section 3, see equation (3.13). The twisting numbers q_j match the electric charges carried by Wilson loops and the intersection numbers p_j match the magnetic charges carried by ’t Hooft loops. Moreover, the identification respects the expected Witten effect on loop operators [18, 19]: when the j^{th} theta angle of the gauge theory is increased by 2π , corresponding to performing a Dehn twist of Σ along γ_j , the Dehn-Thurston parameters change by $(p_j, q_j) \mapsto (p_j, q_j + p_j)$. This identification also matches the intuition one gets from the M-theory constructions, both from looking at M2-branes ending on M5-branes and from the M2-branes in the supergravity background as discussed in appendix A.

The formulation of Dehn’s theorem, as stated, depends on the choice of a pants decomposition (or on the corresponding choice of Fenchel-Nielsen coordinates). It is natural to ask how the data describing non-self-intersecting geodesics changes when the pants decomposition changes. This is essentially the question of S-duality for the field theory: according to Gaiotto’s classification [1], each choice of pants decomposition corresponds to a choice of S-duality frame of the field theory. In geometric terms, these S-dualities are given by the mapping class group of the surface.

The *mapping class group* is the familiar discrete group by which Teichmüller space must be quotiented in order to get the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points; this group relates the various choices of pants decomposition to each other. It is known [25] that the mapping class group is generated by Dehn twists on a small set of geodesic loops. In our setup, some of these will be Dehn twists along the boundaries between pairs of pants (which are easy to interpret in the field theory, as shifts in the theta angles). Others, though, are Dehn twists along loops which cross those boundaries, and these are more difficult to analyze.

Penner [26] gave a general description of the action of the mapping class group on the Dehn-Thurston parameters¹³ $(\mathbb{Z}_{\geq 0})^{3g-3+2n} \times \mathbb{Z}^{3g-3+n}$ as follows: for each element φ of the mapping class group, there is a decomposition K_φ of the vector space $\mathbb{R}^{6g-6+3n}$ into a finite number of cones based at the origin such that on each cone in K_φ , φ acts like an invertible integer matrix. (Following Thurston [23], Penner calls these piecewise-integer-linear (or PIL) transformations.)

¹³Penner’s theorem was explicitly formulated and proven for the case with no punctures. But as he remarks at the end of his paper, the theorem remains true when punctures are allowed, and it is the version with punctures which we have stated here.

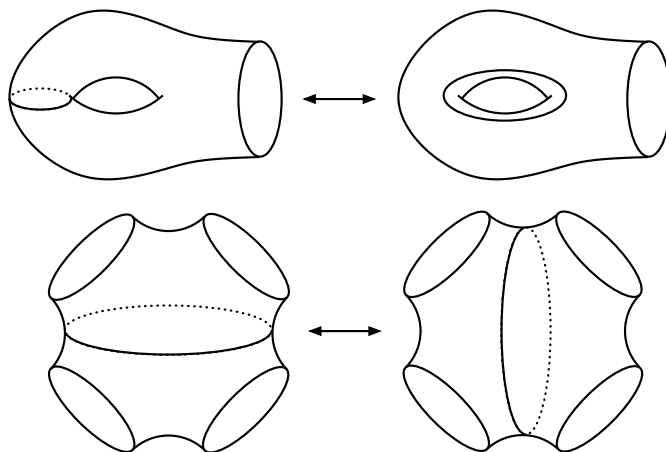


Figure 9. Two types of elementary transformation.

Due to a result of Hatcher and Thurston [27], any two pants decompositions of Σ can be obtained from one another by a sequence of “elementary transformations” of two types, illustrated in figure 9.¹⁴ Penner explicitly computes the PIL transformation on the Dehn-Thurston parameters for each of the two types of elementary transformation, and this is enough to determine it for an arbitrary change of pants decomposition. Working out the transformation rules in specific examples is not too difficult, but Penner’s formulas for general curves and transformations are extremely complicated. We refer the reader to the original paper [26] and the book [24] rather than trying to reproduce those formulas here. However, we will give special cases of Penner’s formulas in some examples in the next subsection.

A change in pants decomposition also induces a change in Fenchel-Nielsen coordinates, that is, in the lengths and twists of the basic geodesics. An explicit coordinate change formula for these has been computed by Okai [29] (again for the two types of elementary transformation), which may be useful in future explorations of S-duality for these theories.

4.1 Examples: checks and predictions

1. A torus with one puncture.

By gluing two pants-legs of a pair of pants together while degenerating the other to a puncture, one obtains a torus with one puncture. As we discussed in section 3.1, this Riemann surface corresponds to $\mathcal{N} = 2^*$ super Yang-Mills as well as $\mathcal{N} = 4$. All geodesics without self-intersection are classified by their intersection number $p_1 \geq 0$ with γ , the twisting number $q_1 \in \mathbb{Z}$ with respect to γ , and the number p_2 of end points at the puncture.

We apply the first type of Hatcher-Thurston transformation (shown in the upper half of figure 9) to obtain $(p'_1, p'_2; q'_1)$ from $(p_1, p_2; q_1)$. Since p_2 represents the number

¹⁴The original formulation of Hatcher and Thurston did not allow punctures, but the result is now known to extend to the punctured case: see [28], for example.

of end-points at the puncture, it is always even and $p'_2 = p_2$. The formula for the other parameters must be divided into cases, according to which type of arc the given charges define on the pair-of-pants, both before and after the transformation. This division into cases gives the piecewise-linear structure of the action of the transformation on the charges. The cases are as follows:

- **Case 1:** If $p_2 > 2p_1$ and $p_1 > |q_1|$ then

$$\begin{aligned} p'_1 &= \frac{p_2}{2} - p_1 + |q_1| \\ q'_1 &= -q_1 \end{aligned}$$

- **Case 2:** If $p_2 > 2p_1$ and $p_1 \leq |q_1|$ then

$$\begin{aligned} p'_1 &= \frac{p_2}{2} - p_1 + |q_1| \\ q'_1 &= -\text{sign}(q_1)p_1 \end{aligned}$$

Note that if $q_1 = 0$ then $p_1 = 0$, so the last line is well-defined.

- **Case 3:** If $p_2 \leq 2p_1$ and $p_2 > 2|q_1|$ then

$$\begin{aligned} p'_1 &= |q_1| \\ q'_1 &= -\text{sign}(q_1) \left(p_1 - \frac{p_2}{2} + |q_1| \right) \end{aligned}$$

where we define $\text{sign}(0) = -1$.

- **Case 4:** If $p_2 \leq 2p_1$ and $p_2 \leq 2|q_1|$ then

$$\begin{aligned} p'_1 &= |q_1| \\ q'_1 &= -\text{sign}(q_1)p_1 \end{aligned}$$

Note that if $q_1 = 0$ then $p_1 = 0$, so the last line is well-defined.

When $p_2 = 0$, we are in case 4, and the formula reproduces the prediction from gauge theory (3.15): it is the standard S-duality transformation from $\text{SL}(2, \mathbb{Z})$, adjusted so that both p_1 and p'_1 are non-negative. All other cases are new predictions for the action of the S-duality group on generalized loop operators with non-trivial bundles of the flavor groups.

The formula above is a special case of Penner's formula [26], but can be obtained directly as follows: the value for the new intersection number p'_1 is obtained by counting the number of times the arcs cross the line segment which becomes a boundary circle after the transformation. The absolute value of the new twisting number q'_1 is then determined by making the formula invertible. To determine the sign of the new twisting number, we need to analyze the diagram of how the arcs change, which we do for each case (in a well-chosen example) below.

In order to explain how to apply the transformation in a particular example, we start with $(p_1, p_2; q_1) = (2, 6; 1)$, depicted on the left side of figure 10. We open the pair

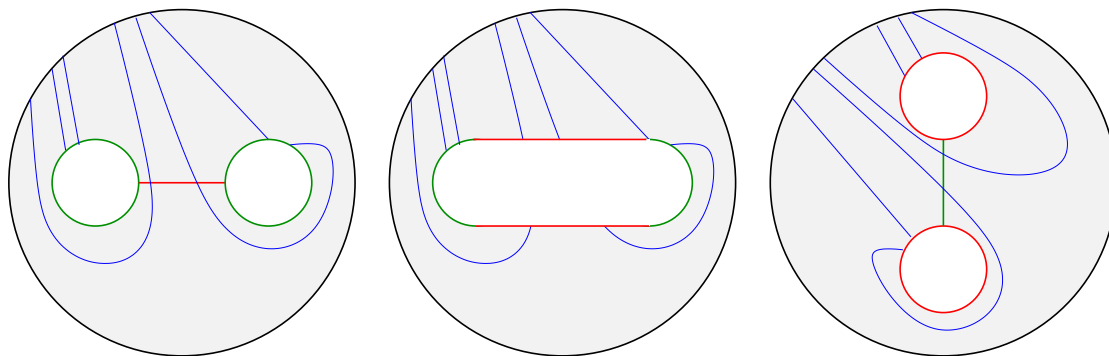


Figure 10. The transformation, step by step.

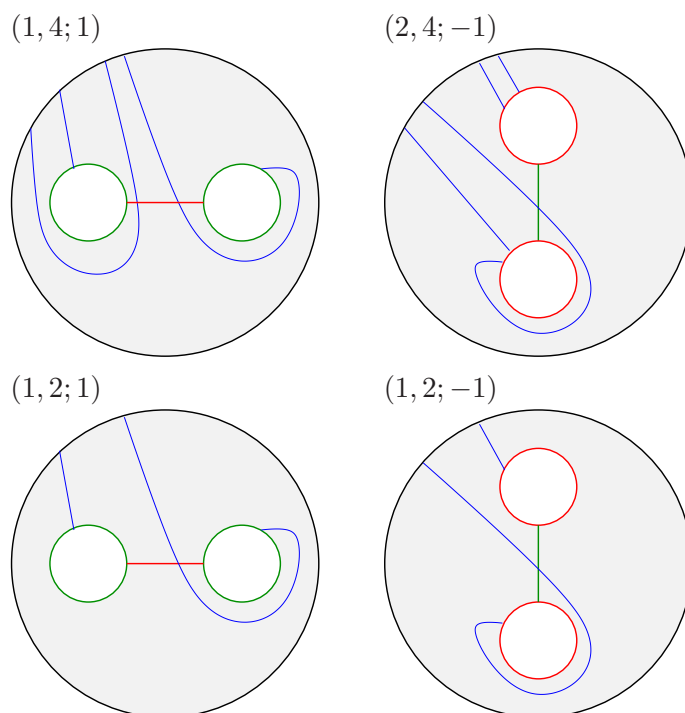


Figure 11. The effect of the transformation on several arcs.

of pants up along the seam, as depicted in the center of figure 10. Then we re-attach the other way, leaving boundary circles in place of the edges of the original seam, as depicted on the right side of figure 10. In this way, we obtain $(p'_1, p'_2; q'_1) = (2, 6, -1)$, and we see that the sign of q'_1 is correct in the above formula.

Similarly, if we choose $(p_1, p_2; q_1) = (1, 4; 1)$ (depicted in the upper left corner of figure 11) and apply the transformation, we find the arc in the upper right corner of figure 11), with $(p'_1, p'_2; q'_1) = (2, 4; -1)$. The end result has $q'_1 = -1$, consistent with the second case of our formula. The third case is simply the inverse of the second case, so we do not need to illustrate that one separately.

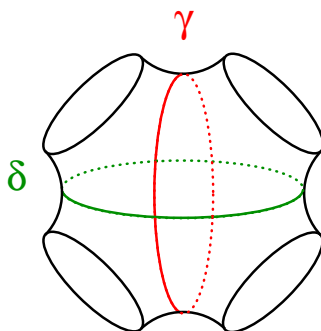


Figure 12. Four-punctured sphere with two meridians.

Finally, if we choose $(p_1, p_2; q_1) = (1, 2; 1)$ (depicted in the lower left corner of figure 11) and apply the transformation, we find the arc in the lower right corner of figure 11), with $(p'_1, p'_2; q'_1) = (1, 2; -1)$. The end result has $q'_1 = -1$, consistent with the fourth case of our formula.

2. A sphere with four-punctures.

The previous example involved the first type of elementary transformation, but we must now consider the second type, whose action on the Dehn-Thurston parameters is much more complicated. If we restrict to closed geodesics, we only need a very special case of the transformation, but in the next example we will need to grapple with the complications.

We begin with the four-punctured sphere, illustrated in figure 12 showing the two meridians γ, δ which represent two possible decompositions into pairs of pants. (We also enlarged the punctures into disks.) A closed non-self-intersecting geodesic with parameters (p, q) (with respect to γ) will meet γ p times, and wind around the surface q times parallel to γ . During each full revolution of winding, the geodesic will meet the other meridian δ two times. Thus, the total intersection with δ is $2|q|$. In other words,

$$\begin{aligned} \#(C \cap \gamma) &= p \\ \#(C \cap \delta) &= 2|q|. \end{aligned} \tag{4.4}$$

Applying the same analysis to the meridian δ (in which the rôles of γ and δ are reversed), we find

$$\begin{aligned} p' &= \#(C \cap \delta) = 2|q| \\ |q'| &= \frac{1}{2} \#(C \cap \gamma) = \frac{p}{2}. \end{aligned} \tag{4.5}$$

By carefully considering the orientation of winding, one can show that the signs of q and q' are opposite. Thus, the S-duality transformation is

$$(p, q) \mapsto \left(2|q|, -\text{sign}(q) \frac{p}{2} \right), \tag{4.6}$$

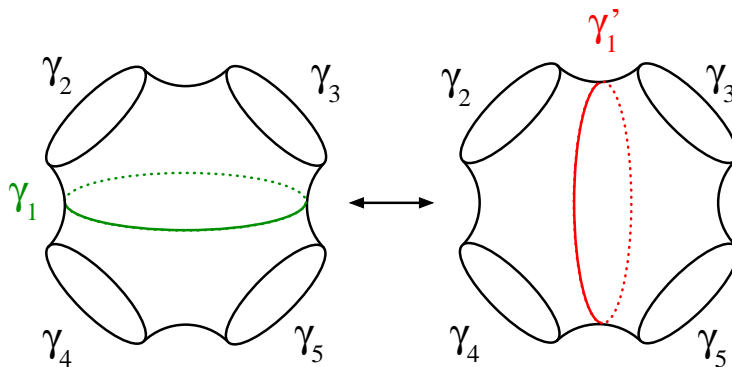


Figure 13. The second type of elementary transformation.

verifying equation (3.19) from the gauge theory (provided that we use the Weyl group to make the magnetic charge positive).

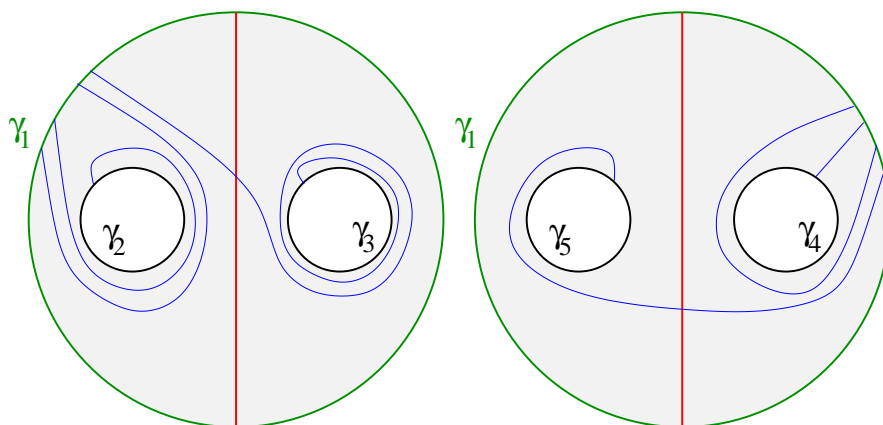
In the same way one can analyze also open geodesics with endpoints at the punctures. This would provide a prediction for the action of S-duality on the loop operators with non-dynamical gauge fields. While this is rather simple to do in specific examples (see also the next subsection), the general case is rather messy to analyze.

3. Genus 2.

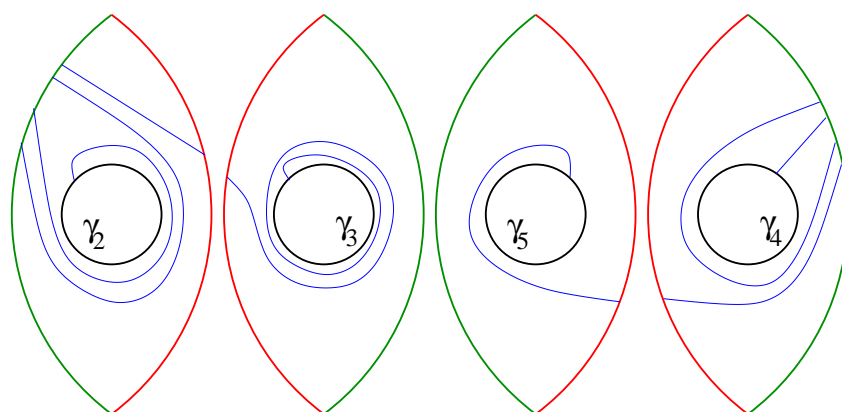
When the Riemann surface Σ has genus two and no punctures, there are two possible decompositions into pairs of pants, represented by the quivers shown in figure 2(b). Both types of Hatcher-Thurston transformation can act here: in the case of the upper quiver from figure 2(b), either of the two pairs of pants in the decomposition is a torus with a disk removed, and the first type of Hatcher-Thurston transformation can be applied. We gave a fairly complete formula for this in example I above, omitting only the second twist parameter (which corresponds to the twist around the loop connecting the two pieces into Σ). The complete formula can be found in the references [24, 26].

The second type of Hatcher-Thurston transformation can also act, exchanging the two quivers in figure 2(b). To describe the action of this transformation on the Dehn-Thurston parameters, we illustrate the transformation again in figure 13, where we have labeled the boundary circles $\gamma_2, \dots, \gamma_5$ as well as the meridians γ_1 and γ'_1 which are used to decompose the surface into two pairs-of-pants, before and after the transformation. For our application to the case of genus two, we identify γ_4 with γ_2 , and γ_5 with γ_3 . Thus, the Hatcher-Thurston transformation moves us from the lower quiver in figure 2(b) to the upper quiver.

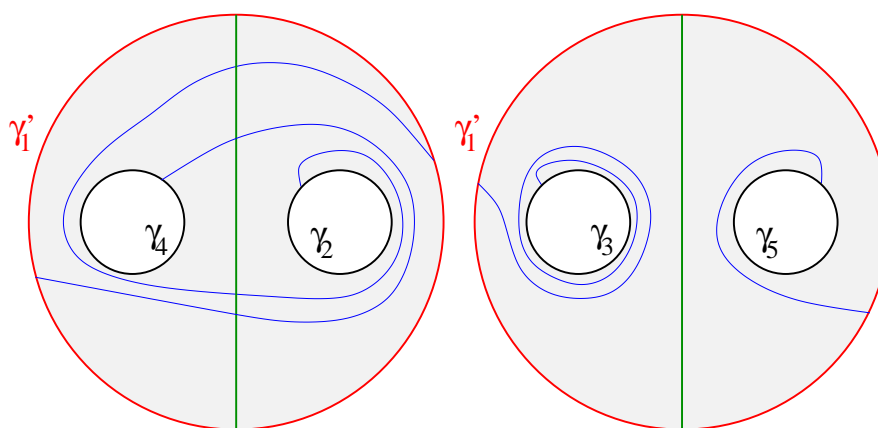
The action of the transformation on any particular geodesic can be found as follows. First cut the surface along γ_1 , yielding a top pair of pants and a bottom pair of pants. On each of these pairs of pants, use Dehn's theorem to put the geodesic into standard form. Next, cut each of the pairs of pants along a portion of γ'_1 , leaving four wedges (one containing each of the other loops $\gamma_2, \dots, \gamma_5$). These wedges can then



The top and bottom pairs of pants.



The four wedges.



The left and right pairs of pants.

Figure 14. The transformation applied to $(4, 1, 1; -2, 1, 2)$.

be reassembled along the corresponding portions of γ_1 , giving a left pair of pants and a right pair of pants, with common boundary curve γ_1' . This is the decomposition

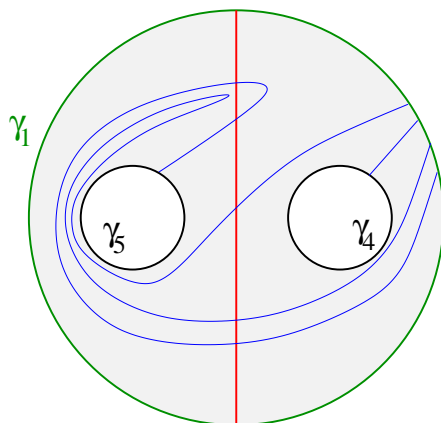


Figure 15. Another view of the bottom pair of pants.

“after” the Hatcher-Thurston transformation, and Dehn’s theorem can again be used to describe the parameters with respect to this new decomposition.

We have illustrated this process in figure 14 for a specific example. We start with the geodesic whose Dehn-Thurston parameters with respect to the lower quiver in figure 2(b) are $(p_1, p_2, p_3; q_1, q_2, q_3) = (4, 1, 1; -2, 1, 2)$. In the top pair of pants, Dehn’s classification implies that we should have one arc of type ℓ_{12} , one of type ℓ_{13} , and one of type ℓ_{11} . (Recall that this latter arc encircles γ_2 .) The twist parameters q_2 and q_3 are also applied in the top pair of pants, giving the curve depicted in the upper left corner of figure 14.

In the bottom pair of pants, Dehn’s classification is similar (since the relevant p_j parameters are identical): we should have one arc of type ℓ_{14} , one of type ℓ_{15} , and one of type ℓ_{11} (which in this case encircles γ_5). We don’t apply the twist parameters q_2 and q_3 since they were already applied to the top pair of pants, but we will apply q_1 here. The clearest depiction of our curve is the one given in figure 15, in which all of this data can be clearly seen. However, for further manipulation, it is better to start with a version in which the arcs have been “pulled taut”, yielding the curve depicted in the upper right corner of figure 14.

Now the procedure is precisely as described above: we cut each of the pairs of pants from the first row of figure 14 in half, giving the four wedges in the second row of figure 14. The attachment of the wedges is such that we can cyclically permute them: the middle two of those wedges form the right pair of pants shown in the lower right of figure 14, while the first and last wedge (joined in the opposite order, after the cyclic permutation) form the left pair of pants shown in the lower left of figure 14.

From the last row of figure 14 we can then read off the Dehn-Thurston parameters with respect to the upper quiver in figure 2(b). The left pair of pants contains one arc of type $\ell_{1'2}$ and another of type $\ell_{1'4}$; it follows that $p'_1 = 2$ and $p'_2 = p'_4 = 1$. Similarly, the right pair of pants contains one arc of type $\ell_{1'3}$ and another of type $\ell_{1'5}$, so that

$p'_1 = 2$ and $p'_3 = p'_5 = 1$. The twist parameters can also be read off of the bottom row: we have $q'_1 = 2$ (seen in the left pair of pants), $q'_3 = 2 - 1$ (seen in the right pair of pants) and $q'_2 = 0$. (Notice that we had to take the sum of the twisting around γ_3 and the twisting around γ_5 in order to compute q'_3 , since those curves are attached on Σ .) The full set of parameter values is $(p'_1, p'_2, p'_3; q'_1, q'_2, q'_3) = (2, 1, 1; 2, 0, 1)$.

This provides a new explicit prediction for the action of the S-duality group on this specific Wilson-'t Hooft operator in the $\mathcal{N} = 2$ conformal theories with gauge group $SU(2)^3$ and no flavor symmetry. Other examples can be worked out in a similar way, and the general formulas can be found in the references [24, 26].

5 Conclusions and discussion

We have classified all 1/2 BPS loop operators in $\mathcal{N} = 2$ generalized quiver conformal field theories which are the IR limit of two coincident M5-branes wrapping an arbitrary Riemann surface with a single type of puncture. Each pants decomposition of the Riemann surface corresponds to a particular duality frame, where we associate an $SU(2)$ gauge group factor to each connected curve along which the surface is cut, and an $SU(2)$ flavor group to each puncture.

In space-time the loops were completely trivial, either a straight line or a circle, and the classification involved only studying the gauge degrees of freedom. One may place an arbitrary 't Hooft loop in any gauge group and as discussed in section 3 it is also natural to allow non-trivial bundles for the flavor groups. In these theories there are hypermultiplets transforming in the fundamental of three groups (gauge or flavor), and in order to couple them consistently to the gauge bundles, one has to impose the usual Dirac quantization condition. The total magnetic charge felt by each such hypermultiplet should be an even integer to ensure that the Dirac string is invisible. We can also include a Wilson loop in an arbitrary representation of each of the gauge group factors. More generally, by combining 't Hooft and Wilson loops, we get dyonic loop operators.

This construction gives two integers for each $SU(2)$ gauge symmetry and one for each flavor $SU(2)$, leading to a set of integers (3.13) subject to the Dirac condition (3.12).

Remarkably, this is exactly the same data as in the classification of homotopy classes of non-self-intersecting curves on Riemann surfaces (possibly disconnected). For each way of cutting the surface into pairs of pants we have an identification of each $SU(2)$ gauge symmetry with a glued pair of pants-legs and an identification of the flavor groups with external punctures. The number of lines crossing a pants-leg is the magnetic charge in that group and the twist which the lines perform there is the electric charge. Dehn's theorem gives precisely the same set of data we have identified in the gauge theory. The exact map is given in table 1.

The S-duality group of the four-dimensional gauge theory is expected to be the mapping class group of the associated Riemann surface. Having a classification of loop operators in terms of the geometrical data allows us to make the following conjecture:

Gauge theory	Riemann surface
SU(2) gauge group	Regular pants-leg
SU(2) flavor symmetry	Degenerate pants-leg (puncture)
T_2 theory (four hypermultiplets)	Pair of pants
Gauge couplings and theta angles	Complex structure moduli
(Generalized) Wilson-'t Hooft loop with weights $(p_j; q_j)$	Non-self-intersecting curve with Dehn-Thurston parameters $(p_j; q_j)$
't Hooft loop in group $SU(2)_j$ with weight p_j	p_j curves crossing γ_j
Wilson loop in group $SU(2)_j$ with charge q_j	Twist q_j on the lines crossing the γ_j , or a disconnected piece of the curve wrapping q_j times the curve γ_j
Non-dynamical monopole field of weight p_j for flavor group $SU(2)_j$	p_j open lines ending at the $[j - (3g - 3)]^{\text{th}}$ puncture
Weyl symmetry	The conditions: $p_j \geq 0$ and if $p_j = 0$ then $q_j \geq 0$
S-duality frame	Choice of a pants decomposition
S-duality group	Mapping class group of the surface

Table 1. Dictionary relating loop operators and geodesics.

The action of the S-duality group on Wilson-'t Hooft operators in the $\mathcal{N} = 2$ theories based on SU(2) gauge factors is given by the action of the mapping class group on homotopy classes of non-self-intersecting curves on the relevant Riemann surface.

Our classification of loop operators in these theories, combined with its identification with Dehn's classification of the curves and Penner's theorem on their transformation rules furnish (in principle) an explicit map for any such gauge theory.

We have illustrated this in several examples. In the case of a single gauge SU(2) and $N_F = 4$ as well as for the $\mathcal{N} = 2^*$ theory, this matches exactly the known expressions. In other cases the general rules are quite complicated, but we demonstrated in several examples how to use Penner's algorithm to analyze the transformation rules and predict the action of the S-duality group on loop operators.

As pointed out in section 4, the classification of homotopy classes of non-self-intersecting curves is identical to the classification of non-self-intersecting geodesics with respect to any fixed hyperbolic metric. Geodesics with respect to hyperbolic metrics are very natural and arise in the large N holographic theory as discussed in appendix A.

Another way to construct these conformal field theories is in terms of M5-branes wrapped on a Riemann surface in space-time. For any Riemann surface with an arbitrary metric there exists a U(1)-invariant Ricci flat metric defined in a neighborhood of the zero-section of the cotangent bundle on the surface [30, 31]. It should be possible to construct

a supersymmetric embedding of an M2-brane ending on the M5-branes. In the six dimensional theory this corresponds to a Wilson surface, and it would be interesting to follow the double dimensional reduction to loop operators in four dimensions. One ends up again with curves on the Riemann surface which now may have an arbitrary metric.¹⁵ Perhaps it is possible to address from this point of view what happens when the curves self-intersect, in particular why this is not allowed on two M5-branes, but is acceptable in the large N limit.

Yet another possible description of this system is in terms of brane webs. One should consider infinitely long (p, q) strings and/or D3-branes ending on the 5-branes whose world volume supports the $\mathcal{N} = 2$ theories [2].

One obvious generalization is to consider Gaiotto's construction of $\mathcal{N} = 2$ conformal field theories based on $SU(N)$ rather than on $SU(2)$ (describing N coincident M5-branes) [1]. There are two difficulties one encounters, the first is just the richness of the construction, with many possible "quiver tails" involving extra groups of rank less than N . The other difficulty with these theories is that the coupling between the different gauge factors is through some strongly interacting theories, of which little is known.

Neglecting possible loop operators that might exist intrinsically in the T_N theory, one can study the conditions that the Wilson-'t Hooft operators constructed from gauge fields must satisfy. An obvious requirement is the Dirac quantization condition. An arbitrary 't Hooft loop in each $SU(N)$ factor is specified by a co-weight μ which is the diagonal matrix $\mu = \text{diag}(m_I - |m|/N)_{I=1}^N$, where m_I are non-negative integers and $|m| = \sum_I m_I$.¹⁶ In each T_N , there is an operator $Q_{a_1 a_2 a_3}$ that transforms in the tri-fundamental representation under the symmetry group $SU(N)_1 \times SU(N)_2 \times SU(N)_3$ of T_N . For $Q_{a_1 a_2 a_3}$ to be single-valued around the Dirac string, we require that the sum $|m^{(1)}| + |m^{(2)}| + |m^{(3)}|$ be divisible by N , where $|m^{(j)}|$ is the quantity $|m|$ associated to $SU(N)_j$ for $j = 1, 2, 3$.

Adding also arbitrary Wilson loops (consistent with the 't Hooft loops) leads to $2N - 2$ integers for each gauge group factor subject to the Weyl symmetry and the Dirac quantization condition. The flavor groups, "quiver tails", and possibly the T_N factors involve extra data. Unfortunately, for $N > 2$ we do not currently have an analogous geometric classification of curves that matches this data. We hope to address this issue in future work [6].

For very large N we can go to strong 't Hooft coupling and study the theory using its M-theory dual, where the loop operators are given by arbitrary geodesics on the Riemann surface with the hyperbolic metric, allowing for self-intersections (see appendix A). The reason self-intersections are allowed is that the M2-brane does not really cross itself, but is separated along another S^1 in the geometry. This is in contrast to the case of $SU(2)$ where crossings were not allowed. For finite N we might expect a "stringy exclusion principle" on this S^1 to somehow restrict the possible crossings on the surface.

¹⁵Note that if the metric on the Riemann surface does not have constant curvature, *i.e.*, is not hyperbolic, the homotopy classes do not have as close a connection to geodesics as they did for hyperbolic metrics. Still, each homotopy class of non-self-intersecting curves can be represented by a shortest-length geodesic which is non-self-intersecting [32], at least in the case without punctures.

¹⁶Using a Weyl transformation we can assume the integers are ordered and then m_I is the number of boxes in the I^{th} row of a Young tableau, which corresponds to a representation of the Langlands dual of the Lie algebra $\mathfrak{su}(N)$. The Dirac condition then means that the sum of N -alities of the three representations is trivial.

Thus far we have only discussed the classification of loop operators, but a very natural question to ask is what their expectation value is. For this purpose one should focus on the case of the loop with a circular geometry in space-time, which still preserves global supersymmetry, but whose expectation value is non-trivial. Note also that the expectation value should depend on the gauge couplings, and hence on the complex structure moduli of the Riemann surface.

Using the supergravity dual discussed in appendix A, the result for large N and strong coupling is very simple to write down. The (Euclidean) M2-brane solution has the geometry $H_2 \times S^1$ (assuming it is connected). The S^1 is related to a geodesic on the Riemann surface, which now has a hyperbolic metric and can be written as H_2/Γ with Γ a discrete subgroup of $SL(2, \mathbb{R})$ (and the two H_2 should not be confused). Any connected geodesic can be related to an element $\gamma \in \Gamma$ and the length of the geodesic is the norm of γ times the unit length on the Riemann surface. Multiplying the M2-brane tension $T_{M2} = 1/4\pi^2$ it appears as the effecting string tension in AdS_5 . For the $H_2 \subset AdS_5$ the standard calculation of the area gives -2π (after canceling a divergent contribution by a boundary term) [33, 34]. In the metric (A.1), the radius of AdS_5 is $2^{1/2}(\pi N)^{1/3}$, and the radius of H_2/Γ is $(\pi N)^{1/3}$. So the vacuum expectation value is

$$\langle L_\gamma \rangle \sim \exp |\gamma| N. \tag{5.1}$$

The gauge coupling appears in $|\gamma|$, since it depends on the moduli of the surface, which are identified with the gauge couplings. In particular, if the surface has a very short geodesic, this corresponds to small gauge coupling and we expect the length of the geodesic to be the gauge coupling g_{YM}^2 , giving

$$\langle L_\gamma \rangle \sim \exp g_{YM}^2 N. \tag{5.2}$$

This dependence is different than for $\mathcal{N} = 4$ SYM, where for the circular Wilson loop

$$\langle L_\gamma \rangle_{\mathcal{N}=4} \sim \exp \sqrt{g_{YM}^2 N}. \tag{5.3}$$

From the gauge theory side, an exact calculation using localization was done by Pestun in [35] which expresses the expectation value of a Wilson loop as a modification of the Gaussian matrix model of [36, 37] by the Nekrasov partition function. This formula applies only for Wilson loops, but if one assumes S-duality, then some 't Hooft or dyonic loops can be transformed into purely electric ones. In fact, if we identify loop operators with geodesics on a Riemann surface, then each non-self-intersecting geodesic is part of a maximal set of disjoint non-self-intersecting geodesics which can be used to cut the surface into pairs of pants and can thus be declared the Wilson loops in an appropriate duality frame. Therefore assuming S-duality, Pestun's formula calculates all circular loop operators for generalized quivers based on $SU(2)$.

In order to check our prediction of the S-duality transformation rules, it would be extremely interesting to have an independent calculation of the expectation values of 't Hooft operators, or more general loop operators. One should be able to repeat the calculation

made in the $\mathcal{N} = 4$ super Yang-Mills case [38, 39] or perhaps generalize the localization argument in [35] to general loop operators.

In fact very recently it was pointed out that in the absence of any loop operators, when Pestun's formula gives the partition function on S^4 , the expression can be reinterpreted as a correlation function in Liouville theory [4]. Using our classification of loop operators it should be possible to extend the correspondence also to the full formula including the expectation value of Wilson loops (and by S-duality to all loop operators).

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A M2-branes as holographic loop operators

To complement the gauge theory analysis in the body of this paper, we consider the analogous problem for gauge theories based on the group $SU(N)$ in the large N limit and at strong coupling, for which there is a dual supergravity description through the AdS/CFT correspondence. It is quite straightforward and unambiguous to construct Wilson, 't Hooft, and dyonic loops in the supergravity theory which is instructive to compare with the gauge theory and topology discussed in the main text. This appendix can be read independently from the rest of the paper, and likewise it can be omitted by the reader interested solely in the gauge theory.

A related gauge theory is $\mathcal{N} = 4$ super Yang-Mills which is dual to type IIB on $AdS_5 \times S^5$, the simplest example of the AdS/CFT correspondence. In this gauge theory, the most symmetric Wilson and 't Hooft operators follow a straight line or a circle in \mathbb{R}^4 . A Wilson loop is described at strong coupling by a fundamental string and an 't Hooft loop by a D1-brane; in both cases they occupy an AdS_2 subspace of AdS_5 . While in the gauge theory Wilson loops and 't Hooft loops are very different objects (in particular, the latter has no simple description as an insertion of operators made of the electric variables), in string theory the difference is just in the choice of brane probe. The distinction between the two gets even smaller when we go back to the M-theory picture, and use a T^2 in place of the hyperbolic Riemann surface. Now both the fundamental string and D1-brane are M2-branes wrapping

some cycle inside T^2 and two directions (AdS_2 or H_2) in the non-compact space. The distinction is simply which cycle they wrap: the one associated with the “11th direction” gives rise to a fundamental string, another to a D1-brane, and a generic one to a (p, q) string.

The situation for more general Riemann surfaces is quite analogous. One considers an M2-brane occupying an $AdS_2 \subset AdS_5$ while following a curve on the Riemann surface.¹⁷ As we show below, to preserve supersymmetry, this curve has to be a geodesic. The electric and magnetic charges of the loop operator under the different gauge groups can be read from the way the geodesic wraps or crosses the corresponding necks on the surface, in exact analogy to the relation between curves and loops described in the main text.

This leads to the identification:

The classification of maximally supersymmetric Wilson-’t Hooft loop operators in $\mathcal{N} = 2$ conformal generalized quiver theories at large N is given by arbitrary geodesics (possibly self-intersecting) on the relevant Riemann surface.

Note that the only distinction from the case of gauge group factors of $SU(2)$ studied in sections 3 and 4 is that for large N self-intersections of the geodesics are allowed.

The gravity duals of the $\mathcal{N} = 2$ conformal theories are obtained as the back-reaction of N M5-branes wrapping a hyperbolic Riemann surface Σ [3]. We restrict our analysis to the case of a Riemann surface without punctures, where the gravitational background is the Maldacena-Nuñez solution [9] given by

$$ds_{11}^2 = (\pi N l_p^3)^{2/3} \frac{W^{1/3}}{2} \left[4ds_{AdS_5}^2 + 2ds_{\Sigma}^2 + 2d\theta^2 + \frac{2}{W} \cos^2 \theta ds_{S^2}^2 + \frac{4}{W} \sin^2 \theta (d\chi + v)^2 \right]. \quad (\text{A.1})$$

Here ds_{Σ}^2 is the metric on Σ with constant scalar curvature -2 , $ds_{S^2}^2$ is the unit metric on the two-sphere and $W = 1 + \cos^2 \theta$. The metric contains a particular one-form v on Σ which we will discuss below.

We now wish to find explicit embeddings of M2-branes that preserve a maximal amount of supersymmetries and represent loop operators. Such loop operators should preserve an $SO(2, 1) \times SO(3) \times SO(3)$ subgroup of the isometry group. We thus assume that the world-volume spans an AdS_2 subspace and sits at the end point $\theta = \pi/2$ of the interval $0 \leq \theta \leq \pi/2$. Then the M2-brane wraps a closed curve \tilde{C} in the three-dimensional space in which the circle parametrized by χ is fibered over Σ . The projection of \tilde{C} defines a closed curve C on Σ .

To study supersymmetry we need eleven-dimensional Killing spinors satisfying the equation

$$\nabla_m \eta + \frac{1}{288} [\Gamma_m^{npqr} - 8\delta_m^n \Gamma^{pqr}] G_{npqr} \eta = 0. \quad (\text{A.2})$$

The metric (A.1) is a special case of the more general metrics found in [40], which characterize $\mathcal{N} = 2$ superconformal field theories in four dimensions. We can thus adapt the Killing spinors obtained there for the background (A.1).¹⁸ There are eight linearly

¹⁷This fact was already mentioned in [3].

¹⁸More precisely, the gravity solutions with four-dimensional $\mathcal{N} = 2$ superconformal symmetries were obtained by analytic continuation from the solutions that describe local operators in six dimensional $(2, 0)$ theories. The latter solutions were found by explicitly constructing Killing spinors.

independent Killing spinors

$$\eta^{\alpha A}, \quad \eta^{c\dot{\alpha}}_A \quad (\alpha = 1, 2, \quad \dot{\alpha} = \dot{1}, \dot{2}, \quad A = 1, 2) \quad (\text{A.3})$$

corresponding to Poincaré supercharges $Q^{\alpha A}$, $\bar{Q}^{\dot{\alpha}}_A$ on the boundary.¹⁹ Here α and $\dot{\alpha}$ are left and right handed spinor indices, while A is for the $SU(2)$ R-symmetry. We use the anti-symmetric tensors ε^{AB} , $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot{\alpha}\dot{\beta}}$ with $\varepsilon^{12} = 1$ to raise and lower indices.

Let us decompose the eleven-dimensional gamma matrices satisfying

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}, \quad m, n = 0, 1, \dots, 10 \quad (\text{A.4})$$

as²⁰

$$\begin{aligned} AdS_5: \quad \Gamma^\mu &= \gamma^\mu \otimes \gamma_{(2)} \otimes \gamma_{(4)}, \quad \mu = 0, 1, 2, 3, 4, \\ S^2: \quad \Gamma^{5,6} &= 1_4 \otimes \gamma^{5,6} \otimes \gamma_{(4)}, \\ \Sigma, \chi, \theta: \quad \Gamma^i &= 1_4 \otimes 1_2 \otimes \gamma^i, \quad i = 7, 8, 9, 10. \end{aligned} \quad (\text{A.5})$$

$\gamma_{(2)}$ and $\gamma_{(4)}$ are chirality matrices with eigenvalues ± 1 . The Killing spinors can be expressed in terms of lower dimensional Killing spinors as

$$\eta^{\alpha A} = e^{\tilde{\lambda}/2} \psi^\alpha \otimes \left[(1 + \gamma_{(2)} \otimes \gamma_{(4)}) \cdot \chi^A \otimes e^{\frac{\zeta}{2} \gamma_9} e^{i\chi/2} \epsilon_0 \right], \quad (\text{A.6})$$

$$\eta^{c\dot{\alpha}}_A = e^{\tilde{\lambda}/2} \psi^c_{\dot{\alpha}} \otimes \left[(1 - \gamma_{(2)} \otimes \gamma_{(4)}) \cdot \chi_A \otimes e^{-\frac{\zeta}{2} \gamma_9} e^{-i\chi/2} \gamma_7 \epsilon_0 \right], \quad (\text{A.7})$$

where ζ is determined by $y = -e^{3\tilde{\lambda}} \sinh \zeta$ in terms of the quantities y and $\tilde{\lambda}$ defined in [3]. The fixed four-component spinor ϵ_0 satisfies

$$(i\gamma^9 \gamma_{(4)} + 1) \epsilon_0 = (1 - i\gamma^7 \gamma^8) \epsilon_0 = 0. \quad (\text{A.8})$$

We have introduced AdS_5 Killing spinors ψ^α satisfying the equation

$$D_m \psi^\alpha = \frac{1}{2} \gamma_m \psi^\alpha, \quad m = 0, 1, \dots, 4, \quad (\text{A.9})$$

and $\psi^c_{\dot{\alpha}}$ satisfying

$$D_m \psi^c_{\dot{\alpha}} = -\frac{1}{2} \gamma_m \psi^c_{\dot{\alpha}}, \quad m = 0, 1, \dots, 4. \quad (\text{A.10})$$

The two S^2 Killing spinors χ^A satisfy

$$D_m \chi^A = \frac{i}{2} \gamma_m \chi^A, \quad m = 5, 6. \quad (\text{A.11})$$

By writing the AdS_5 metric in terms of Poincaré coordinates as

$$ds^2 = \frac{dx^\mu dx_\mu + dz^2}{z^2} \quad (\text{A.12})$$

¹⁹Note that $\alpha, \dot{\alpha}$, and A label eight spinors, each of which has 32 components. Also c indicates charge conjugation and is not an index.

²⁰To keep equations simple, we keep implicit the distinction between coordinate and frame indices.

with $\mu = 0, 1, 2, 3$ and $z \equiv x^4$, the AdS_5 Killing spinors can be explicitly given as

$$\psi^\alpha = z^{-1/2}\psi_{(0)}^\alpha, \quad \psi_{\dot{\alpha}}^c = z^{-1/2}\psi_{(0)\dot{\alpha}}^c, \quad (\text{A.13})$$

where $\psi_{(0)}^\alpha$ and $\psi_{(0)\dot{\alpha}}^c$ are constant spinors

$$\psi_{(0)}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_{(0)}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_{(0)\dot{1}}^c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_{(0)\dot{2}}^c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A.14})$$

We represent the AdS_5 gamma matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.15})$$

The linear combination of supercharges $Q = \xi_{\alpha A} Q^{\alpha A} + \bar{\xi}_{\dot{\alpha}}^A \bar{Q}_{\dot{\alpha} A}$ on the boundary corresponds to the spinor

$$\eta = \xi_{\alpha A} \eta^{\alpha A} + \bar{\xi}_{\dot{\alpha}}^A \eta^{c\dot{\alpha}}_A. \quad (\text{A.16})$$

in the bulk. In our ansatz for the world-volume of an M2-brane, we have

$$x^1 = x^2 = x^3 = 0, \quad \theta = \pi/2. \quad (\text{A.17})$$

The condition for supersymmetry [41]

$$\frac{1}{3!} \varepsilon^{abc} \Gamma_{mnp} \partial_a X^m \partial_b X^n \partial_c X^p \eta = -\eta \quad (\text{A.18})$$

reduces, in the static gauge, to

$$\Gamma_{04} \Gamma_{\tilde{C}} \eta = -\eta, \quad (\text{A.19})$$

where 0 and 4 are the AdS_2 directions, and

$$\Gamma_{\tilde{C}} = t^i \Gamma_i \quad (\text{A.20})$$

is the gamma matrix in the direction of the curve \tilde{C} . Let us decompose the tangent vector t with respect to the orthonormal frame:

$$t = t^7 e_7 + t^8 e_8 + t^9 e_9. \quad (\text{A.21})$$

By collecting terms with the same eigenvalues of γ^4 in (A.19), one finds that

$$-\bar{\xi}_{\dot{\alpha} A} \gamma_7 \epsilon_0 + (\bar{\sigma}_0 \xi_A)^{\dot{\alpha}} t^i \gamma_i e^{i\chi} \epsilon_0 = 0, \quad \xi_{\alpha A} \epsilon_0 + (\sigma_0 \bar{\xi}_A)_\alpha e^{-i\chi} t^i \gamma_i \gamma_7 \epsilon_0 = 0, \quad (\text{A.22})$$

or equivalently

$$\begin{aligned} \left[-\bar{\xi}_{\dot{\alpha} A} + (\bar{\sigma}_0 \xi_A)^{\dot{\alpha}} (t^7 - it^8 + t^9 \gamma_9 \gamma_7) e^{i\chi} \right] \gamma_7 \epsilon_0 &= 0, \\ \left[\xi_{\alpha A} + (\sigma_0 \bar{\xi}_A)_\alpha e^{-i\chi} (t^7 + it^8 + t^9 \gamma_9 \gamma_7) \right] \epsilon_0 &= 0, \end{aligned} \quad (\text{A.23})$$

Since $\gamma_9\gamma_7$ changes the eigenvalue of $\gamma_7\gamma_8$ for which ϵ_0 is an eigenvector, we need that

$$0 = t^9 \propto \dot{\chi} + v_7 \dot{x}^7 + v_8 \dot{x}^8, \tag{A.24}$$

where the dot indicates the derivative with respect to the proper length. Then (A.23) implies that

$$\bar{\xi}^\alpha_A = e^{i\chi_0} (\bar{\sigma}_0 \xi_A)^\alpha, \tag{A.25}$$

$$\arg(t^7 + it^8) = \chi - \chi_0 \tag{A.26}$$

with χ_0 being a constant. It is natural to introduce a complex coordinate $w = re^{i\beta}$ on Σ , in terms of which the metric and the one-form v are given by

$$ds_\Sigma^2 = 4 \frac{dr^2 + r^2 d\beta^2}{(1-r^2)^2} = \frac{4dw d\bar{w}}{(1-|w|^2)^2}, \quad v = \frac{2r^2 d\beta}{1-r^2}. \tag{A.27}$$

Then any real vector field $V = V^i \partial_i$ ($i = 7, 8$) with unit norm takes the form

$$V = \frac{1-|w|^2}{2} e^{i\varphi} \partial_w + c.c., \tag{A.28}$$

and its covariant derivative is given by

$$DV = D_i V^j dx^j \otimes \partial_j = ie^{i\varphi} \frac{1-|w|^2}{2} \left(d\varphi + \frac{2r^2}{1-r^2} d\beta \right) \otimes \partial_w + c.c. \tag{A.29}$$

If we take the tangent vector $\dot{x}^i \partial_i = \dot{w} \partial_w + \overline{\dot{w}} \partial_{\bar{w}}$ of the curve C on Σ to be of unit norm, (A.26) implies that $\arg(\dot{w}) = \chi - \chi_0$. Then from (A.24) together with (A.29) we find that $\dot{x}^i \partial_i$ is covariantly constant, *i.e.*,

$$\ddot{x}^i + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0 \quad (i, j, k = 7, 8). \tag{A.30}$$

Thus the curve C has to be a geodesic. The condition (A.24) implies that \tilde{C} is also a geodesic in the total space of the circle fibration over Σ .

Conversely, one can lift any closed geodesic C on Σ , to a closed geodesic \tilde{C} on the total space of the circle fibration by setting

$$\chi = \arg(\dot{w}) + \chi_0 \tag{A.31}$$

for some constant χ_0 . The relation (A.31) between the orientation of the tangent vector of C and the position χ on the circle implies that even if the geodesic C self-intersects, the M2-brane world-volume does not. This relation amounts to the vanishing of the last term in (A.1) (v is exactly canceled by $d\chi$) which guarantees that the lengths of C and \tilde{C} are equal.

We conclude that for an arbitrary geodesic C , an M2-brane wrapping the uplift \tilde{C} and the AdS_2 subspace of AdS_5 is half BPS. The supercharges preserved by the M2-brane are determined by (A.25).

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